§11.5 Chain Rule

#3. $w = xe^{\frac{y}{z}}$, $x = t^2$, $y = 1-t$, $z = 1+2t$, 求 $\frac{dw}{dt}$

sol:

$$t \xmapsto{\vec{r}} (\underset{t^2}{\overset{\|}{x}}, \underset{1-t}{\overset{\|}{y}}, \underset{1+2t}{\overset{\|}{z}}) \xmapsto{f} xe^{\frac{y}{z}} = w$$

$$\frac{dw}{dt} = w_x \frac{dx}{dt} + w_y \frac{dy}{dt} + w_z \frac{dz}{dt}$$

$$= e^{\frac{y}{z}} \cdot 2t + \frac{x}{z} e^{\frac{y}{z}} \cdot (-1) - \frac{xy}{z^2} e^{\frac{y}{z}} \times 2$$

$$= e^{\frac{y}{z}} \left(2t - \frac{x}{z} - \frac{2xy}{z^2} \right)$$ #

(代入 $x = t^2$, $y = 1-t$, $z = 1+2t$)

= 全換成 t ......

#7. $z = e^r \cos\theta$, $r = st$, $\theta = \sqrt{s^2+t^2}$, 求 $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

sol:

$$(s, t) \xrightarrow{\vec{R}} (\underset{st}{\overset{\|}{r}}, \underset{\sqrt{s^2+t^2}}{\overset{\|}{\theta}}) \xrightarrow{f} e^r \cos\theta = z$$

$$\frac{\partial z}{\partial s} = z_r r_s + z_\theta \theta_s = e^r \cos\theta \cdot t - e^r \sin\theta \frac{s}{\sqrt{s^2+t^2}}$$ #

$$\frac{\partial z}{\partial t} = z_r r_t + z_\theta \theta_t = e^r \cos\theta \cdot s - e^r \sin\theta \frac{t}{\sqrt{s^2+t^2}}$$ #

#9. $z = f(x, y)$ is diff. $x = g(t)$, $y = h(t)$, $g(3) = 2$, $g'(3) = 5$, $h(3) = 7$, $h'(3) = -4$, $f_x(2,7) = 6$, $f_y(2,7) = -8$, find $\left.\frac{dz}{dt}\right|_{t=3} = ?$

sol:

$$\underset{3}{\overset{\downarrow}{t}} \longrightarrow (\underset{g(t)}{\overset{\|}{x}}, \underset{h(t)}{\overset{\|}{y}}) \xrightarrow{f} f(x, y) = z$$

$g(3) = 2$ $h(3) = 7$

當 $t = 3$ 時 $(x, y) = (g(3), h(3)) = (2, 7)$

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

$$\left.\frac{dz}{dt}\right|_{t=3} = f_x(2,7)\, g'(3) + f_y(2,7)\, h'(3)$$

$$= 6 \times 5 + (-8)(-4) = 30 + 32 = 62$$ #

§11.5 Chain Rule

#11. Suppose f is diff. of x and y, and $g(u,v) = f(e^u + \sin v,\ e^u + \cos v)$. Use the table to calculate $g_u(0,0)$ and $g_v(0,0)$.

sol:

$$(u,v) \longrightarrow (\boxed{e^u+\sin v},\ e^u+\cos v) \longrightarrow f(x,y) = z$$

$$x = e^u+\sin v, \quad y = e^u+\cos v$$

$$\text{At } (0,0) \longrightarrow (1,\ 2)$$

$$g$$

$$g_u = f_x\, x_u + f_y\, y_u = f_x(e^u) + f_y(e^u) = e^u[f_x + f_y]$$

$$g_v = f_x\, x_v + f_y\, y_v = f_x(\cos v) + f_y(-\sin v)$$

At $(u,v) = (0,0)$, $x = 1$, $y = 2$

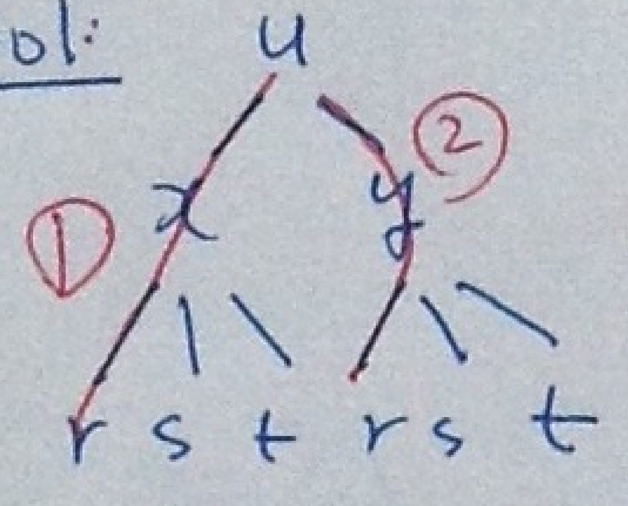
$$g_u(0,0) = [f_x(1,2) + f_y(1,2)] \underset{查表}{=} 2 + 5 = 7$$

$$g_v(0,0) = f_x(1,2) = 2$$

{#13, #15} Use a tree diagram to write out the Chain Rule

#13. $u = f(x,y)$, $x = x(r,s,t)$, $y = y(r,s,t)$

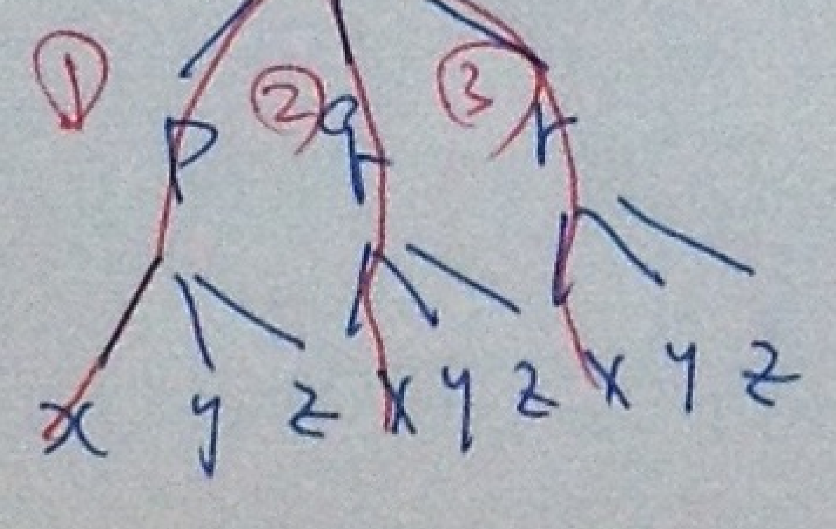
sol:



$$\begin{cases} u_r = \underbrace{u_x\, x_r}_{①} + \underbrace{u_y\, y_r}_{②} \\ u_s = u_x\, x_s + u_y\, y_s \\ u_t = u_x\, x_t + u_y\, y_t \end{cases}$$

#15. $v = f(p,q,r)$, $p = p(x,y,z)$, $q = q(x,y,z)$, $r = r(x,y,z)$

sol:



$$\begin{cases} v_x = \underbrace{v_p\, p_x}_{①} + \underbrace{v_q\, q_x}_{②} + \underbrace{v_r\, r_x}_{③} \\ v_y = v_p\, p_y + v_q\, q_y + v_r\, r_y \\ v_z = v_p\, p_z + v_q\, q_z + v_r\, r_z \end{cases}$$

§11.5 Chain Rule

#19. $R=\ln(u^2+v^2+w^2)$, $u=x+2y$, $v=2x-y$, $w=2xy$;

R_x, R_y when $(x,y)=(1,1)$

sol:

$$\begin{array}{ccccc} (x,y) & \longmapsto & (u, & v, & w) & \longmapsto & \ln(u^2+v^2+w^2)=R \\ & & \| & \| & \| & & \\ & & \boxed{x+2y} & \boxed{2x-y} & 2xy & & \\ (1,1) & \longmapsto & (3, & 1, & 2) & & \end{array}$$

$$R_x = R_u u_x + R_v v_x + R_w w_x$$

$$= \frac{2u}{u^2+v^2+w^2}\times 1 + \frac{2v}{u^2+v^2+w^2}\times 2 + \frac{2w}{u^2+v^2+w^2}\times 2y$$

$$= \frac{2u+4v+4yw}{u^2+v^2+w^2}$$

同理, $R_y = \dfrac{4u-2v+4wx}{u^2+v^2+w^2}$

當 $x=y=1 \Rightarrow u=3, v=1, w=2$ 代入上式

$$R_x(1,1) = \frac{6+4+8}{3^2+1+2^2} = \frac{18}{14} = \underline{\frac{9}{7}}\ \#,\quad R_y(1,1) = \underline{\frac{9}{7}}\ \#$$

#21. $u=x^2+yz$, $x=pr\cos\theta$, $y=pr\sin\theta$, $z=p+r$;

$\frac{\partial u}{\partial p}$, $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial \theta}$ when $p=2, r=3, \theta=0$

sol:

$$\begin{array}{ccccc} (p,r,\theta) & \longrightarrow & (x, & y, & z) & \longrightarrow & x^2+yz=u \\ & & \| & \| & \| & & \\ & & pr\cos\theta & pr\sin\theta & p+r & & \\ (2,3,0) & \longmapsto & (6, & 0, & 5) & & \end{array}$$

$$u_p = u_x x_p + u_y y_p + u_z z_p = 2xr\cos\theta + zr\sin\theta + y$$

$$u_r = u_x x_r + u_y y_r + u_z z_r = 2xp\cos\theta + zp\sin\theta + y$$

$$u_\theta = u_x x_\theta + u_y \theta_y + u_z z_\theta = -2xpr\sin\theta + zpr\cos\theta$$

when $p=2, r=3, \theta=0 \Rightarrow x=6, y=0, z=5$

so $u_p=36$, $u_r=24$, $u_\theta=30$ at $(p,r,\theta)=(2,3,0)$

\#

§11.6. Chain Rule

#23. Use formula for implicit fct to find $\frac{dy}{dx}$

$\sqrt{xy} = 1 + x^2 y$

sol: Let $u(x,y) = (xy)^{\frac{1}{2}} - x^2 y$

Then $u_x = \frac{1}{2}\left(\frac{y}{x}\right)^{\frac{1}{2}} - 2xy$, $u_y = \frac{1}{2}\left(\frac{x}{y}\right)^{\frac{1}{2}} - x^2$

By formula, $\frac{dy}{dx} = -\frac{u_x}{u_y} = \frac{4xy - \left(\frac{y}{x}\right)^{\frac{1}{2}}}{\left(\frac{x}{y}\right)^{\frac{1}{2}} - 2x^2}$ # or $\frac{4(xy)^{\frac{3}{2}} - y}{x - 2x^2\sqrt{xy}}$ #

#27. Use formula for implicit fct to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$x - z = \tan^{-1}(yz)$

sol: Let $u(x,y,z) = x - z - \tan^{-1}(yz)$

Then $u_x = 1$, $u_y = \frac{-z}{1+y^2z^2}$, $u_z = -1 - \frac{y}{1+y^2z^2} = \frac{-1-y^2z^2-y}{1+y^2z^2}$

By formula,

$$\frac{\partial z}{\partial x} = -\frac{u_x}{u_z} = \frac{1+y^2z^2}{1+y^2z^2+y} \text{ \#} \quad \text{and}$$

$$\frac{\partial z}{\partial y} = -\frac{u_y}{u_z} = \frac{-z}{1+y^2z^2+y} \text{ \#}$$

#37. Assum that all the given fcts are diff.

$z = f(x,y)$, where $x = r\cos\theta$ and $y = r\sin\theta$,

(a) find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ and (b) show that

$(z_x)^2 + (z_y)^2 = (z_r)^2 + \frac{1}{r^2}(z_\theta)^2$ ……(一階偏導)

sol (a): By cham Kue, $z_r = z_x\cos\theta + z_y\sin\theta$

$z_\theta = z_x(-r\sin\theta) + z_y(r\cos\theta)$ #

pf (b): $(z_r)^2 = (z_x)^2\cos^2\theta + (z_y)^2\sin\theta + 2z_x z_y\cos\theta\sin\theta$

会消掉

$(z_\theta)^2 = (z_x)^2(r^2\sin^2\theta) + (z_y)^2(r^2\cos^2\theta) - 2z_x z_y r^2\cos\theta\sin\theta$

$\Rightarrow (z_r)^2 + \frac{1}{r^2}\cdot(z_\theta)^2 = (z_x)^2 + (z_y)^2$ □

§11.5. Chain Rule (求2階偏導)

#43. $z = f(x,y)$, where $x = r^2+s^2$, $y = 2rs$, find $\dfrac{\partial^2 z}{\partial r \partial s}$

sol:

$$\begin{tikzcd}(r,s) \arrow[r] \arrow[rr, bend right] & (x,y) \arrow[r, "f"] & f(x,y)=z\end{tikzcd}$$

$x = r^2+s^2$, $y = 2rs$

$$\begin{cases} x_r = 2r \\ x_s = 2s \end{cases} \qquad \begin{cases} y_r = 2s \\ y_s = 2r \end{cases}$$

$$\frac{\partial^2 z}{\partial r \partial s} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial s}\right) = \frac{\partial}{\partial r}\left[z_x\, 2s + z_y\, 2r\right]$$

$$= \left[z_{xx}\, 2r\cdot 2s + z_{xy}\, 2s \cdot 2s\right] + z_{yx}(2r)^2 + z_{yy}\, 2s\cdot 2r + z_y \cdot 2$$

$$= 4rs\,(f_{xx}+f_{yy}) + (4r^2+4s^2)\,f_{xy} + 2z_y \quad ※$$

#45. $z = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$, show that

$$z_{xx} + z_{yy} = z_{rr} + \frac{1}{r^2} z_{\theta\theta} + \frac{1}{r} z_r$$

Pf:

$$\begin{tikzcd}(r,\theta) \arrow[r] \arrow[rr, bend right] & (x,y) \arrow[r, "f"] & f(x,y)=z\end{tikzcd}$$

$x = r\cos\theta$, $y = r\sin\theta$

$$\begin{cases} x_r = \cos\theta \\ x_\theta = -r\sin\theta \end{cases} \qquad \begin{cases} y_r = \sin\theta \\ y_\theta = r\cos\theta \end{cases}$$

$$z_r = z_x \cos\theta + z_y \sin\theta$$

對r微　　對r微

$$z_{rr} = z_{xx}\cos^2\theta + \underbrace{z_{xy}\cos\theta\sin\theta + z_{yx}\cos\theta\sin\theta}_{\text{可合併}} + z_{yy}\sin^2\theta$$

$$z_\theta = z_x(-r\sin\theta) + z_y(r\cos\theta)$$

$$z_{\theta\theta} = z_{xx}(r^2\sin^2\theta) + z_{xy}(-r^2\cos\theta\sin\theta) + z_x(-r\cos\theta)$$
$$+ z_{yx}(-r^2\cos\theta\sin\theta) + z_{yy}(r^2\cos^2\theta) + z_y(-r\sin\theta)$$

整理

$$z_{rr} = \cos^2\theta\, z_{xx} + \sin^2\theta\, z_{yy} + 2\cos\theta\sin\theta\, z_{xy}$$

$$\frac{1}{r^2} z_{\theta\theta} = \sin^2\theta\, z_{xx} + \cos^2\theta\, z_{yy} - 2\cos\theta\sin\theta\, z_{xy} - \frac{1}{r}\cos\theta\, z_x - \frac{1}{r}\sin\theta\, z_y$$

Thus

$$z_{rr} + \frac{1}{r^2} z_{\theta\theta} = z_{xx} + z_{yy} - \frac{1}{r}\big(\underbrace{\cos\theta\, z_x + \sin\theta\, z_y}_{= z_r}\big)$$

$$\Rightarrow z_{xx} + z_{yy} = z_{rr} + \frac{1}{r^2} z_{\theta\theta} + \frac{1}{r} z_r \qquad \square$$